Hyperbolic Kähler-Ricci Flow

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Abstract

In this paper, the author has considered the hyperbolic Kähler-Ricci flow introduced by Kong and Liu [11], that is, the hyperbolic version of the famous Kähler-Ricci flow. The author has explained the derivation of the equation and calculated the evolutions of various quantities associated to the equation including the curvatures. Particularly on Calabi-Yau manifolds, the equation can be simplified to a scalar hyperbolic Monge-Ampère equation which is just the hyperbolic version of the corresponding one in Kähler-Ricci flow.

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§1 Introduction

Recently, a new flow on Riemannian manifolds is introduced by Kong and Liu([11], [10], [5]):

$$\frac{\partial^2}{\partial t^2} g = -2Rc.$$

This is the hyperbolic version of the famous Ricci flow. They studied its short-time existence in compact case, derived evolutions of curvatures which have wave character. A dissipative flow was also considered in [6]. One remarkable result is [12], in which the authors proved that on compact surfaces with metric depending on one space variable, the flow has a global solution if the initial velocity is large enough. So given an initial metric, the geometric surgery can be replaced by choosing suitable velocity to allow the long-time existence. However, the evolutions of Riemann curvature, Ricci curvature and the scalar curvature are very complicated which contain the first-order derivative of time that is hard to control. In this paper, I consider its complex version which is also introduced by Kong and Liu in [11]:

$$\frac{\partial^2}{\partial t^2} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}.$$

As we will see on Calabi-Yau manifolds, it can be reduced to a scalar (complex) hyperbolic Monge-Ampère equation which is much easier to handle. The flow considered here can also be regarded as the hyperbolic version of Kăhler-Ricci flow:

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}.$$

In the past several decades, nonlinear partial differential equations played important role in differential geometry. For example, the resolutions of Calabi's conjecture [19] and Poincaré's conjecture [1] [13] [15] [16] [17] are due to the methods of geometric analysis and the elliptic and parabolic type of equations are intensively studied. However, the hyperbolic equations have been ignored for a long time. For the elliptic and parabolic equations, we have the powerful tool the maximum principle which cannot be applied to hyperbolic ones. This lack of practical tool may explain the reason why the research of hyperbolic equations on manifolds are not as active as the other two types. However, Perelman [15] [16] [17] introduced several powerful tools into the study of flow like

the energy functional, the monotonicity formula, and the space-time geometry. In the Euclidean space, these methods are available for hyperbolic equations. So I believe we can refer to Perelman's methods to study nonlinear hyperbolic equations on manifolds.

Hyperbolic equations are important in physics especially in general relativity. The famous Einstein's equation has been studied for a long time. In fact, as illustrated in [11], the hyperbolic version of Ricci flow is closely related to Einstein's equation. So I hope, by studying the wave character of manifolds through hyperbolic equations, we can better understand the basic structure of the universe. There is already some work on wave equations on Lorentzian manifolds which catches more interest from Physicists. Although wave equations on Riemannian manifolds are also considered, the results are few and have a more analytic style that its influence on curvature are not considered. In my opinion, analyzing wave equations on manifolds from a more geometric viewpoint so that it may give more help to understand the geometry and topology of manifolds.

Let $(\mathcal{M}^n, g, J)$ be a complete Kähler manifold, where g is the metric depending on time and J is a fixed complex structure. The hyperbolic Kähler-Ricci flow is the following evolution equation

$$\frac{\partial^2}{\partial t^2} g_{i\bar{j}} = -R_{i\bar{j}} \tag{1.1}$$

for a family of Kähler metrics $g_{i\bar{j}}(t)$ on $\mathcal{M}^n$. A natural and fundamental problem is the short-time existence and uniqueness theorem of (1.1). We will see as in the case of Kähler-Ricci flow, the evolution equation of metrics can be simplified to a scalar evolution equation. In the present paper, I also derive the corresponding wave equations for various geometric quantities.

Theorem 1.1. *Let $(\mathcal{M}^n, g^0_{i\bar{j}}(z))$ be a compact Calabi-Yau manifold with $g^0_{i\bar{j}}(z)$ a Kähler metric. Then there exists a constant $h > 0$ such that the initial value problem*

$$\begin{cases} \frac{\partial^2}{\partial t^2} g_{i\bar{j}}(z,t) = -R_{i\bar{j}}(z,t) \\ g_{i\bar{j}}(z,0) = g^0_{i\bar{j}}(z), \quad \frac{\partial}{\partial t} g_{i\bar{j}}(z,0) = g^1_{i\bar{j}}(z), \end{cases}$$

has a unique smooth solution $g_{i\bar{j}}$ on $\mathcal{M} \times [0,h]$ and $g_{i\bar{j}}(z,t)$ remains Kähler for any $t > 0$ as long as the solution exists and the initial velocity must satisfy $[\frac{\partial}{\partial t}\omega(z,0)] = 0$ where $\omega(z,0)$ is the Kähler form of $g_{i\bar{j}}(z,0)$.

Similar to Kähler-Ricci flow, I will derive the corresponding wave equations for the curvatures.

Theorem 1.2. *Under the hyperbolic Kähler-Ricci flow (1.1), the Riemannian curvature tensor, Ricci curvature and scalar curvature satisfy the evolution equations under a unitary coordinate*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{i\bar{j}k\bar{l}} &= \triangle_R R_{i\bar{j}k\bar{l}} + R_{i\bar{\alpha}\beta\bar{l}}R_{\alpha\bar{j}k\bar{\beta}} - R_{i\bar{\alpha}k\bar{\beta}}R_{\alpha\bar{j}\beta\bar{l}} + R_{i\bar{j}\beta\bar{\alpha}}R_{\alpha\bar{\beta}k\bar{l}} \\ &\quad -\frac{1}{2}(R_{i\bar{\alpha}}R_{\alpha\bar{j}k\bar{l}} + R_{\alpha\bar{j}}R_{i\bar{\alpha}k\bar{l}} + R_{k\bar{\alpha}}R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}}R_{i\bar{j}k\bar{\alpha}}) \\ &\quad +2g^{p\bar{q}}\nabla_k\left(\frac{\partial}{\partial t}g_{i\bar{q}}\right)\nabla_{\bar{l}}\left(\frac{\partial}{\partial t}g_{p\bar{j}}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{i\bar{j}} &= \triangle_R R_{i\bar{j}} + R_{i\bar{j}k\bar{l}}R_{l\bar{k}} - R_{i\bar{k}}R_{k\bar{j}} - 2\left\langle\frac{\partial}{\partial t}g_{k\bar{l}}, \frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}}\right\rangle \\ &\quad +2R_{i\bar{j}k\bar{l}}\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{r\bar{m}}g^{n\bar{l}}g^{k\bar{s}} \\ &\quad +2g^{p\bar{q}}g^{k\bar{l}}\nabla_k\left(\frac{\partial}{\partial t}g_{i\bar{q}}\right)\nabla_{\bar{l}}\left(\frac{\partial}{\partial t}g_{p\bar{j}}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R &= \triangle R + |R_{\alpha\bar{\beta}}|^2 + \triangle\left|\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}\right|^2 - 2\left\langle\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}, \frac{\partial}{\partial t}R_{\alpha\bar{\beta}}\right\rangle \\ &\quad +2R_{k\bar{l}}\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{r\bar{m}}g^{n\bar{l}}g^{k\bar{s}}.\end{aligned}$$

Here $\triangle_R = \frac{1}{2}(\nabla_\beta\nabla_{\bar{\beta}} + \nabla_{\bar{\beta}}\nabla_\beta)$.

Particularly, on Calabi-Yau manifolds, the flow can be reduced to a hyperbolic Monge-Ampère equation:

$$\begin{cases} \frac{\partial^2\varphi}{\partial t^2}(z,t) = \log\det g_{\alpha\bar{\beta}}(z,t) - \log\det g_{\alpha\bar{\beta}}(z,0) - f_0 \\ \varphi(z,0) = \varphi_0(z), \qquad \frac{\partial\varphi}{\partial t}(z,0) = \varphi_1(z)\end{cases} \tag{3.2}$$

where

$$g_{\alpha\bar{\beta}}(z,t) \doteqdot g_{\alpha\bar{\beta}}(z,0) + \frac{\partial^2\varphi}{\partial z^\alpha\partial\bar{z}^\beta}(z,t).$$

The paper is organized as follows. In Section 2, I will review briefly some basics in Kähler geometry. In Section 3, I will discuss the derivation and basic facts about hyperbolic Kähler-Ricci flow in detail. In Section 4, evolutions of various geometric quantities including the curvatures will be derived.

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§2 Basic Kähler geometry

I state some basic facts about Kähler geometry that will be used in this paper. For a thorough study of Kähler geometry, [9] [14] [18] [20] are good choices. Let $\mathcal{M}^n$ be an n-dimensional compact Kähler manifold. A Hermitian metric is given by

$$g = \sum g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta.$$

Its associated Kähler form is

$$\omega = -\frac{1}{2} Img = \frac{\sqrt{-1}}{2} \sum g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

The Kähler condition requires that ω is a closed positive global $(1,1)$-form.

The Riemann curvature is locally given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l},$$

and has the symmetries

$$R_{i\bar{j}k\bar{l}} = R_{k\bar{l}i\bar{j}} = -R_{\bar{j}ik\bar{l}} = -R_{i\bar{j}\bar{l}k}.$$

The Ricci curvature is defined by

$$R_{i\bar{j}} = g^{k\bar{l}} R_{i\bar{j}k\bar{l}},$$

and locally we have

$$R_{i\bar{j}} = -\frac{\partial^2 \log \det(g_{k\bar{l}})}{\partial z^i \partial \bar{z}^j}.$$

The associated Ricci form is

$$\rho = \frac{\sqrt{-1}}{2} \sum R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

It is a real closed $(1,1)$-form and represents the first Chern class. The scalar curvature is

$$R = g^{i\bar{j}} R_{i\bar{j}}.$$

Given a Kähler metric, there are some other important quantities. The nonzero Christoffel symbols are given by

$$\Gamma^k_{ij} = \sum_{l=1}^n g^{k\bar{l}} \frac{\partial g_{i\bar{l}}}{\partial z^j} \qquad and \qquad \Gamma^{\bar{k}}_{\bar{i}\bar{j}} = \sum_{l=1}^n g^{\bar{k}l} \frac{\partial g_{l\bar{i}}}{\partial \bar{z}^j}.$$

The volume form is

$$d\mu = \frac{\omega^n}{n!} = (\sqrt{-1})^n \det(g_{i\bar{j}}) dz^1 \wedge d\bar{z}^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n.$$

The average scalar curvature is

$$r = \frac{\int_{\mathcal{M}^n} R d\mu}{\int_{\mathcal{M}^n} d\mu}.$$

Finally I give an important lemma which is a consequence of the Hodge decomposition theorem.

Lemma 2.1($\partial\bar{\partial}-$Lemma). *Let $\mathcal{M}^n$ be a compact Kähler manifold. If a is an exact real (1,1)-form, then there exists a real-valued function ψ on $\mathcal{M}^n$ such that $\sqrt{-1}\partial\bar{\partial}\psi = a$. That is,*

$$\frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}\psi = a_{\alpha\bar{\beta}},$$

where $a = \sqrt{-1}a_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ and $\overline{a_{\alpha\bar{\beta}}} = a_{\beta\bar{\alpha}}$.

Proof. This is a standard result in the theory of Kähler manifolds, see [20] for example.

□

§3 Hyperbolic Kähler-Ricci flow

In this section I will discuss the hyperbolic Kähler-Ricci flow and its equivalent formulation as a single hyperbolic Monge-Ampère equation. For the Kähler-Ricci flow theory, please see [4]. I also suggest reader to read Hamilton's papers for more details about Ricci flow [7] [8]. The book [3] is also a good choice.

Given a compact complex manifold $\mathcal{M}^n$, consider the hyperbolic Kähler-Ricci flow equation

$$\frac{\partial^2}{\partial t^2}g_{i\bar{j}} = -R_{i\bar{j}},$$

for a 1-parameter family of Kähler metrics, which is obtained from the Ricci flow by dropping the factor of 2 with respect to the convention. On a Calabi-Yau manifold, the first Chern class vanishes, so $[\rho] = 0$. Under this condition, from the $\partial\bar{\partial} - Lemma$, the Ricci tensor has a potential, i.e.

$$R_{\alpha\bar{\beta}} = \nabla_\alpha \nabla_{\bar{\beta}} f,$$

where $f(z,t)$ is a function defined on the manifold.

Assume

$$g_{\alpha\bar{\beta}}(z,t) = g_{\alpha\bar{\beta}}(z,0) + \nabla_\alpha \nabla_{\bar{\beta}}\varphi(z,t),$$

where $\varphi(z,t)$ is a function on the manifold, since we have

$$R_{\alpha\bar{\beta}}(t) = -\nabla_\alpha \nabla_{\bar{\beta}} \log\det\left(g_{\gamma\bar{\delta}}(0) + \frac{\partial^2 \varphi(t)}{\partial z^\gamma \partial \bar{z}^\delta}\right).$$

Hence

$$\begin{aligned}
\nabla_\alpha \nabla_{\bar{\beta}} \frac{\partial^2 \varphi(z,t)}{\partial t^2} &= \frac{\partial^2}{\partial t^2} g_{\alpha\bar{\beta}}(z,t) = -R_{\alpha\bar{\beta}}(z,t) = (R_{\alpha\bar{\beta}}(z,0) - R_{\alpha\bar{\beta}}(z,t)) - R_{\alpha\bar{\beta}}(z,0) \\
&= \nabla_\alpha \nabla_{\bar{\beta}} \log \frac{\det(g^0_{\gamma\bar{\delta}} + \nabla_\gamma \nabla_{\bar{\delta}} \varphi)}{\det g^0_{\gamma\bar{\delta}}} - \nabla_\alpha \nabla_{\bar{\beta}} f_0.
\end{aligned}$$

So on Calabi-Yau manifolds, the hyperbolic Kähler-Ricci flow equation is equivalent to the following hyperbolic (scalar) complex Monge-Ampère equation due to the maximum principle for compact manifolds:

$$\frac{\partial^2}{\partial t^2}\varphi = \log \frac{\det\left(g^0_{\gamma\bar{\delta}} + \frac{\partial^2 \varphi}{\partial z^\gamma \partial \bar{z}^\delta}\right)}{\det g^0_{\gamma\bar{\delta}}} - f_0 + c_1(t)$$

for some function of time $c_1(t)$ satisfying the compatibility condition

$$\int_{\mathcal{M}^n} \left(\frac{\partial^2 \varphi}{\partial t^2} - f_0\right) d\mu = \exp(c_1(t)) Vol(\mathcal{M}^n),$$

here the Volume $Vol(\mathcal{M}^n)$ is under the metric $g_{\alpha\bar{\beta}}(z,t)$. Further, by a time-dependent translation of $\varphi(z,t)$, we can drop the factor $c_1(t)$.

Further we have

$$\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}(z,0) = \nabla_\alpha \nabla_{\bar{\beta}} \frac{\partial}{\partial t} \varphi(z,0),$$

So we can let $[\frac{\partial}{\partial t}\omega(z,0)] = 0$.

From the hyperbolicity and short time existence and uniqueness of our hyperbolic Monge-Ampére equation, we have proved the Theorem 1.1.

Next let us consider the normalized flow which is also studied in [11] in the real case. Choose the normalization factor $\varphi = \varphi(t)$ (Note: this has nothing to do with the φ above),

$$\tilde{g}_{i\bar{j}} = \varphi^2 g_{i\bar{j}}$$

such that

$$\int_{\mathcal{M}^n} d\widetilde{V} = 1,$$

and choose a new time parameter

$$\tilde{t} = \int_{\mathcal{M}^n} \varphi(t) dt.$$

Noting that for the normalized metric $\tilde{g}_{i\bar{j}}$, we have

$$\tilde{R}_{i\bar{j}} = R_{i\bar{j}}, \qquad \tilde{R} = \frac{1}{\varphi^2} R, \qquad \tilde{r} = \frac{1}{\varphi^2} r.$$

Thus

$$\frac{\partial \tilde{g}_{i\bar{j}}}{\partial \tilde{t}} = \varphi \frac{\partial g_{i\bar{j}}}{\partial t} + 2 \frac{d\varphi}{dt} g_{i\bar{j}},$$

$$\begin{aligned} \frac{\partial^2 \tilde{g}_{i\bar{j}}}{\partial \tilde{t}^2} &= \frac{\partial^2 g_{i\bar{j}}}{\partial t^2} + 3 \left(\frac{d}{dt} \log \varphi \right) \frac{\partial g_{i\bar{j}}}{\partial t} + 2 \left(\frac{d}{dt} \log \varphi \right) \left(\frac{d}{dt} \log \frac{d\varphi}{dt} \right) g_{i\bar{j}} \\ &= -\tilde{R}_{i\bar{j}} + 3 \frac{1}{\varphi} \left(\frac{d}{dt} \log \varphi \right) \frac{\partial \tilde{g}_{i\bar{j}}}{\partial \tilde{t}} + 2 \frac{1}{\varphi^2} \left(\frac{d}{dt} \log \varphi \right) \left(\frac{d}{dt} \log \frac{d\varphi}{dt} - 3 \frac{d}{dt} \log \varphi \right) \tilde{g}_{i\bar{j}} \\ &= -\tilde{R}_{i\bar{j}} + a \frac{\partial \tilde{g}_{i\bar{j}}}{\partial \tilde{t}} + b \tilde{g}_{i\bar{j}}, \end{aligned}$$

where a and b are certain functions of t.

Next let us consider the following hyperbolic system

$$\begin{cases} \frac{\partial^2 \varphi}{\partial t^2}(z,t) = \log \frac{\det g_{\alpha\bar{\beta}}(z,t)}{\det g_{\alpha\bar{\beta}}(z,0)} - f_0 \\ \varphi(z,0) = \varphi_0(z), \qquad \frac{\partial \varphi}{\partial t}(z,0) = \varphi_1(z) \end{cases} \tag{3.1}$$

where

$$g_{\alpha\bar{\beta}}(z,t) \doteqdot g_{\alpha\bar{\beta}}(z,0) + \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}(z,t).$$

Let

$$v(z,t) \doteqdot -\frac{\partial \varphi}{\partial t}(z,t).$$

We get

$$\frac{\partial^2 v}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}\right) = -\frac{\partial}{\partial t} g_{\alpha\bar{\beta}}.$$

So

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} v &= -\frac{\partial}{\partial t}\left(\frac{\partial^2 \varphi}{\partial t^2}\right) \\
&= -\frac{\partial}{\partial t}(\log\det g_{\alpha\bar{\beta}}) + \frac{\partial}{\partial t}(\log\det g^0_{\alpha\bar{\beta}}) - \frac{\partial}{\partial t} f_0 \\
&= -g^{\alpha\bar{\beta}} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} \\
&= g^{\alpha\bar{\beta}} \frac{\partial^2 v}{\partial z^\alpha \partial \bar{z}^\beta} \\
&= \triangle v,
\end{aligned}$$

thus we have

$$\frac{\partial^2}{\partial t^2} v = \triangle v.$$

Note that the Laplacian operator here is time-dependent so the equation is genuinely nonlinear.

§4 Evolutions of geometric quantities

The hyperbolic Kähler-Ricci flow is a hyperbolic evolution equation on the metrics. The evolution of the metrics implies nonlinear wave equations for the Riemannian curvature tensor $R_{i\bar{j}k\bar{l}}$, the Ricci curvature $R_{i\bar{j}}$ and the scalar curvature R which I will derive. I also derive the evolutions of some other quantities in this section. For the evolutions which are similar to those in this section associated to Kähler-Ricci flow, please cite [4].

Let $\mathcal{M}^n$ be an n-dimentional compact Kähler manifold. Let us consider the hyperbolic Kähler-Ricci flow on $\mathcal{M}^n$,

$$\frac{\partial^2}{\partial t^2} g_{i\bar{j}}(z,t) = -R_{i\bar{j}}(z,t).$$

Proposition 4.1(Riemannian curvature tensor). *In a unitary frame,*

$$\begin{aligned}\frac{\partial^2}{\partial t^2} R_{i\bar{j}k\bar{l}} &= \triangle_R R_{i\bar{j}k\bar{l}} + R_{i\bar{\alpha}\beta\bar{l}} R_{\alpha\bar{j}k\bar{\beta}} - R_{i\bar{\alpha}k\bar{\beta}} R_{\alpha\bar{j}\beta\bar{l}} + R_{i\bar{j}\beta\bar{\alpha}} R_{\alpha\bar{\beta}k\bar{l}} \\ &\quad -\frac{1}{2}(R_{i\bar{\alpha}} R_{\alpha\bar{j}k\bar{l}} + R_{\alpha\bar{j}} R_{i\bar{\alpha}k\bar{l}} + R_{k\bar{\alpha}} R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}} R_{i\bar{j}k\bar{\alpha}}) \\ &\quad +2g^{p\bar{q}} \nabla_k \left(\frac{\partial}{\partial t} g_{i\bar{q}}\right) \nabla_{\bar{l}} \left(\frac{\partial}{\partial t} g_{p\bar{j}}\right)\end{aligned}$$

The above formula also holds in arbitrary holomorphic coordinates if repeated indices are contracted via the metric. I.e., if $R_{i\bar{\alpha}\beta\bar{l}} R_{\alpha\bar{j}k\bar{\beta}}$ is replaced by $g^{\gamma\bar{\alpha}} g^{\beta\bar{\delta}} R_{i\bar{\alpha}\beta\bar{l}} R_{\gamma\bar{j}k\bar{\delta}}$, etc.

In the Kähler case, it is sometimes convenient to compute locally in holomorphic coordinates in terms of ordinary derivatives. The following lemma translates these ordinary derivatives to the covariant derivatives, [4].

Lemma 4.2(Relation between ordinary and covariant derivatives). *Let η be a closed (1,1)-form. Locally it is represented by $\eta_{\alpha\bar{\beta}}$, which is Hermitian symmetric. Let $\eta_{\alpha\bar{\beta},\gamma\bar{\delta}}$ denote the covariant derivatives and $\eta_{\alpha\bar{\beta}\gamma\bar{\delta}}$ denote $\frac{\partial^2}{\partial z^\gamma \partial z^{\bar{\delta}}} \eta_{\alpha\bar{\beta}}$. Then at the center x of normal holomorphic coordinates*

$$\begin{aligned}\eta_{\gamma\bar{\delta},\alpha\bar{\beta}} &= \eta_{\gamma\bar{\delta}\alpha\bar{\beta}} + \eta_{s\bar{\delta}} R_{\alpha\bar{\beta}\gamma\bar{s}}, \\ \eta_{\gamma\bar{\delta},\bar{\beta}\alpha} &= \eta_{\gamma\bar{\delta}\bar{\beta}\alpha} + \eta_{\gamma\bar{s}} R_{\alpha\bar{\beta}s\bar{\delta}}.\end{aligned}$$

Proof.

$$\begin{aligned}\eta_{\gamma\bar{\delta},\alpha\bar{\beta}} &= \nabla_{\bar{\beta}} \nabla_\alpha \eta_{\gamma\bar{\delta}} = \partial_{\bar{\beta}} \nabla_\alpha \eta_{\gamma\bar{\delta}} - \bar{\Gamma}^{\varepsilon}_{\beta\delta} \nabla_\alpha \eta_{\gamma\bar{\varepsilon}} \\ &= \partial_{\bar{\beta}} (\partial_\alpha \eta_{\gamma\bar{\delta}} - \Gamma^{\varepsilon}_{\alpha\gamma} \eta_{\varepsilon\bar{\delta}}) \\ &= \partial_{\bar{\beta}} \partial_\alpha \eta_{\gamma\bar{\delta}} - (\partial_{\bar{\beta}} \Gamma^{\varepsilon}_{\alpha\gamma}) \eta_{\varepsilon\bar{\delta}} - \Gamma^{\varepsilon}_{\alpha\gamma} (\partial_{\bar{\beta}} \eta_{\varepsilon\bar{\delta}}) \\ &= \eta_{\gamma\bar{\delta}\alpha\bar{\beta}} + R^{\varepsilon}_{\alpha\bar{\beta}\gamma} \eta_{\varepsilon\bar{\delta}}.\end{aligned}$$

In the above calculation, we use the fact that η is closed so $\nabla_\alpha \eta_{\gamma\bar{\varepsilon}} = \partial_{\bar{\beta}} \eta_{\varepsilon\bar{\delta}} = 0$. The

second formula is the conjugate of the first.

□

Proof of Proposition 4.1. Recall that

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l}$$

This implies that, in a normal holomorphic coordinate system centered at any given point

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} R_{i\bar{j}k\bar{l}} &= -\frac{\partial^2}{\partial z^k \partial \bar{z}^l}\left(\frac{\partial^2}{\partial t^2} g_{i\bar{j}}\right) + 2g^{p\bar{q}} \nabla_k \left(\frac{\partial}{\partial t} g_{i\bar{q}}\right) \nabla_{\bar{l}} \left(\frac{\partial}{\partial t} g_{p\bar{j}}\right) \\
&= \frac{\partial^2}{\partial z^k \partial \bar{z}^l} R_{i\bar{j}} + 2g^{p\bar{q}} \nabla_k \left(\frac{\partial}{\partial t} g_{i\bar{q}}\right) \nabla_{\bar{l}} \left(\frac{\partial}{\partial t} g_{p\bar{j}}\right) \\
&= \nabla_k \nabla_{\bar{l}} R_{i\bar{j}} - R_{i\bar{\alpha}} R_{\alpha\bar{j}k\bar{l}} + 2g^{p\bar{q}} \nabla_k \left(\frac{\partial}{\partial t} g_{i\bar{q}}\right) \nabla_{\bar{l}} \left(\frac{\partial}{\partial t} g_{p\bar{j}}\right).
\end{aligned}$$

The proposition follows from the following two identities:

$$\begin{aligned}
\nabla_k \nabla_{\bar{l}} R_{i\bar{j}} &= \nabla_k \nabla_{\bar{l}} R_{i\bar{j}\beta\bar{\beta}} = \nabla_k \nabla_{\bar{\beta}} R_{i\bar{j}\beta\bar{l}} \\
&= \nabla_{\bar{\beta}} \nabla_{\beta} R_{i\bar{j}k\bar{l}} - R_{i\bar{\alpha}k\bar{\beta}} R_{\alpha\bar{j}\beta\bar{l}} + R_{\alpha\bar{j}k\bar{\beta}} R_{i\bar{\alpha}\beta\bar{l}} \\
&\quad - R_{\beta\bar{\alpha}k\bar{\beta}} R_{i\bar{j}\alpha\bar{l}} + R_{\alpha\bar{l}k\bar{\beta}} R_{i\bar{j}\beta\bar{\alpha}},
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\bar{\beta}} \nabla_{\beta} R_{i\bar{j}k\bar{l}} &= \nabla_{\beta} \nabla_{\bar{\beta}} R_{i\bar{j}k\bar{l}} + R_{i\bar{\alpha}\beta\bar{\beta}} R_{\alpha\bar{j}k\bar{l}} - R_{\alpha\bar{j}\beta\bar{\beta}} R_{i\bar{\alpha}k\bar{l}} \\
&\quad + R_{k\bar{\alpha}\beta\bar{\beta}} R_{i\bar{j}\alpha\bar{l}} - R_{\alpha\bar{l}\beta\bar{\beta}} R_{i\bar{j}k\bar{\alpha}}.
\end{aligned}$$

Recall that $\triangle_R = \frac{1}{2}(\nabla_\beta \nabla_{\bar{\beta}} + \nabla_{\bar{\beta}} \nabla_\beta)$.

□

Lemma 4.3. *We will need the following two basic formulas under hyperbolic Kähler-Ricci flow:*

$$\frac{\partial}{\partial t} g^{k\bar{l}} = -g^{k\bar{s}} \left(\frac{\partial}{\partial t} g_{r\bar{s}}\right) g^{r\bar{l}},$$

$$\frac{\partial^2}{\partial t^2}g^{k\bar{l}} = R_{r\bar{s}}g^{k\bar{s}}g^{r\bar{l}} + 2\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{k\bar{s}}g^{r\bar{m}}g^{n\bar{l}}.$$

Corollary 4.4(Ricci curvature). *The Ricci curvature satisfies the following in a unitary frame:*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{i\bar{j}} &= \triangle_R R_{i\bar{j}} + R_{i\bar{j}k\bar{l}}R_{l\bar{k}} - R_{i\bar{k}}R_{k\bar{j}} - 2\left\langle\frac{\partial}{\partial t}g_{k\bar{l}}, \frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}}\right\rangle \\ &\quad +2R_{i\bar{j}k\bar{l}}\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{r\bar{m}}g^{n\bar{l}}g^{k\bar{s}} \\ &\quad +2g^{p\bar{q}}g^{k\bar{l}}\nabla_k\left(\frac{\partial}{\partial t}g_{i\bar{q}}\right)\nabla_{\bar{l}}\left(\frac{\partial}{\partial t}g_{p\bar{j}}\right)\end{aligned}$$

Proof.

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{i\bar{j}} &= \frac{\partial^2}{\partial t^2}(g^{k\bar{l}}R_{i\bar{j}k\bar{l}}) \\ &= g^{k\bar{l}}\left(\frac{\partial^2}{\partial t^2}R_{i\bar{j}k\bar{l}}\right) + R_{i\bar{j}k\bar{l}}\left(\frac{\partial^2}{\partial t^2}g^{k\bar{l}}\right) + 2\left(\frac{\partial}{\partial t}g^{k\bar{l}}\right)\left(\frac{\partial}{\partial t}R_{i\bar{j}k\bar{l}}\right).\end{aligned}$$

Put the evolutions of $R_{i\bar{j}k\bar{l}}$ and $g^{k\bar{l}}$ in, we get the result.

□

Proposition 4.5(Evolution of R). *The scalar curvature R evolves by*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R &= \triangle R + |R_{\alpha\bar{\beta}}|^2 + \triangle\left|\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}\right|^2 - 2\left\langle\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}, \frac{\partial}{\partial t}R_{\alpha\bar{\beta}}\right\rangle \\ &\quad +2R_{k\bar{l}}\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{r\bar{m}}g^{n\bar{l}}g^{k\bar{s}}.\end{aligned}$$

Proof. First we have

$$\frac{\partial}{\partial t}\log\det g = g^{\alpha\bar{\beta}}\frac{\partial}{\partial t}g_{\alpha\bar{\beta}},$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\log\det g &= -\left|\frac{\partial}{\partial t}g\right|^2 + g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial t^2}g_{\alpha\bar{\beta}} \\ &= -\left|\frac{\partial}{\partial t}g\right|^2 - R.\end{aligned}$$

Thus

$$\frac{\partial^2}{\partial t^2}R_{\alpha\bar{\beta}} = -\nabla_\alpha\nabla_{\bar{\beta}}\left(\frac{\partial^2}{\partial t^2}\log\det g\right) = \nabla_\alpha\nabla_{\bar{\beta}}R + \nabla_\alpha\nabla_{\bar{\beta}}\left|\frac{\partial}{\partial t}g\right|^2.$$

So finally

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R &= -2\left\langle\frac{\partial}{\partial t}g_{\gamma\bar{\delta}}, \frac{\partial}{\partial t}R_{\gamma\bar{\delta}}\right\rangle + g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial t^2}R_{\alpha\bar{\beta}} + \left(\frac{\partial^2}{\partial t^2}g^{\alpha\bar{\beta}}\right)R_{\alpha\bar{\beta}} \\ &= \triangle R + |R_{\alpha\bar{\beta}}|^2 + \triangle\left|\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}\right|^2 - 2\left\langle\frac{\partial}{\partial t}g_{\alpha\bar{\beta}}, \frac{\partial}{\partial t}R_{\alpha\bar{\beta}}\right\rangle \\ &\quad +2R_{k\bar{l}}\left(\frac{\partial}{\partial t}g_{n\bar{m}}\right)\left(\frac{\partial}{\partial t}g_{r\bar{s}}\right)g^{r\bar{m}}g^{n\bar{l}}g^{k\bar{s}}\end{aligned}$$

□

Next I present the evolution equations of the Christoffel symbols and the volume form.

Proposition 4.6. *The christoffel symbols and the volume form evolve respectively by*

$$\frac{\partial^2}{\partial t^2}\Gamma^\gamma_{\alpha\beta} = -g^{\gamma\bar{\delta}}\nabla_\alpha R_{\beta\bar{\delta}} + 2\left(\frac{\partial}{\partial t}g^{\gamma\bar{\delta}}\right)\nabla_\alpha\left(\frac{\partial}{\partial t}g_{\beta\bar{\delta}}\right),$$

$$\frac{\partial^2}{\partial t^2}d\mu = \left[-R + \left(g^{\alpha\bar{\beta}}\frac{\partial g_{\alpha\bar{\beta}}}{\partial t}\right)^2 - \left|\frac{\partial g_{\alpha\bar{\beta}}}{\partial t}\right|^2\right]d\mu.$$

Proof. Recall that

$$\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\bar{\delta}} \left(\frac{\partial}{\partial z^{\alpha}} g_{\beta\bar{\delta}} + \frac{\partial}{\partial z^{\beta}} g_{\alpha\bar{\delta}} - \frac{\partial}{\partial \bar{z}^{\delta}} g_{\alpha\beta} \right) = g^{\gamma\bar{\delta}} \frac{\partial}{\partial z^{\alpha}} g_{\beta\bar{\delta}},$$

thus

$$\frac{\partial}{\partial t} \Gamma^{\gamma}_{\alpha\beta} = g^{\gamma\bar{\delta}} \frac{\partial}{\partial z^{\alpha}} \left(\frac{\partial}{\partial t} g_{\beta\bar{\delta}} \right) + \left(\frac{\partial}{\partial t} g^{\gamma\bar{\delta}} \right) \frac{\partial}{\partial z^{\alpha}} g_{\beta\bar{\delta}}.$$

So finally in normal coordinate

$$\begin{aligned} \frac{\partial^2}{\partial t^2} \Gamma^{\gamma}_{\alpha\beta} &= 2\left(\frac{\partial}{\partial t} g^{\gamma\bar{\delta}} \right) \frac{\partial}{\partial z^{\alpha}} \left(\frac{\partial}{\partial t} g_{\beta\bar{\delta}} \right) + g^{\gamma\bar{\delta}} \frac{\partial}{\partial z^{\alpha}} \left(\frac{\partial^2}{\partial t^2} g_{\beta\bar{\delta}} \right) + \left(\frac{\partial^2}{\partial t^2} g^{\gamma\bar{\delta}} \right) \left(\frac{\partial}{\partial z^{\alpha}} g_{\beta\bar{\delta}} \right) \\ &= -g^{\gamma\bar{\delta}} \nabla_{\alpha} R_{\beta\bar{\delta}} + 2\left(\frac{\partial}{\partial t} g^{\gamma\bar{\delta}} \right) \nabla_{\alpha} \left(\frac{\partial}{\partial t} g_{\beta\bar{\delta}} \right). \end{aligned}$$

The evolution of $d\mu$ follows from the quantities

$$\frac{\partial}{\partial t} \log \det g = g^{\alpha\bar{\beta}} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}}$$

and

$$\frac{\partial^2}{\partial t^2} \det g_{\alpha\bar{\beta}} = \frac{\partial}{\partial t} (\det g) \cdot g^{\alpha\bar{\beta}} \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} - (\det g) \left| \frac{\partial}{\partial t} g_{\alpha\bar{\beta}} \right|^2 + (\det g) \cdot g^{\alpha\bar{\beta}} (-R_{\alpha\bar{\beta}}).$$

□

Proposition 4.7(potential of Ricci curvature). *On compact Kähler manifolds, the potential f of Ricci curvature ($R_{\alpha\bar{\beta}} = \nabla_{\alpha} \nabla_{\bar{\beta}} f$) satisfies:*

$$\frac{\partial^2}{\partial t^2} f = \triangle f + \left| \frac{\partial}{\partial t} g \right|^2 + c(t).$$

Proof.

$$\begin{aligned} \nabla_{\alpha} \nabla_{\bar{\beta}} \left(\frac{\partial^2}{\partial t^2} f \right) &= \frac{\partial^2}{\partial t^2} \nabla_{\alpha} \nabla_{\bar{\beta}} f = \frac{\partial^2}{\partial t^2} R_{\alpha\bar{\beta}} \\ &= \nabla_{\alpha} \nabla_{\bar{\beta}} \left(R + \left| \frac{\partial}{\partial t} g \right|^2 \right) \end{aligned}$$

$$= \quad \nabla_\alpha \nabla_{\bar\beta}\left(\triangle f + \left|\frac{\partial}{\partial t}g\right|^2\right),$$

so from the maximum principle of compact manifolds,

$$\frac{\partial^2}{\partial t^2}f = \triangle f + \left|\frac{\partial}{\partial t}g\right|^2 + c(t).$$

□

By a suitable time-dependent translation of f, we can drop the factor $c(t)$.

Proposition 4.8. *On compact Kähler manifolds, the evolutions of* $\int_{\mathcal{M}^n} Rd\mu$ *and* r *are:*

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{\mathcal{M}^n} Rd\mu \quad &= \quad \int_{\mathcal{M}^n} R\left[\left(g^{\alpha\bar\beta}\frac{\partial g_{\alpha\bar\beta}}{\partial t}\right)^2 - \left|\frac{\partial g_{\alpha\bar\beta}}{\partial t}\right|^2\right]d\mu + 2\int_{\mathcal{M}^n}\frac{\partial}{\partial t}R\frac{\partial}{\partial t}d\mu \\
&\quad -2\int_{\mathcal{M}^n}\left\langle\frac{\partial}{\partial t}g_{\alpha\bar\beta}, \frac{\partial}{\partial t}R_{\alpha\bar\beta}\right\rangle d\mu \\
&\quad +2\int_{\mathcal{M}^n} R_{k\bar l}\left(\frac{\partial}{\partial t}g_{n\bar m}\right)\left(\frac{\partial}{\partial t}g_{r\bar s}\right)g^{r\bar m}g^{n\bar l}g^{k\bar s}d\mu,
\end{aligned}$$

$$\begin{aligned}
\frac{d^2 r}{dt^2} \quad &= \quad r^2 + \frac{\left(\frac{d^2}{dt^2}\int_{\mathcal{M}^n} Rd\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)} + \frac{2\left(\frac{d}{dt}\int_{\mathcal{M}^n} d\mu\right)^2\left(\int_{\mathcal{M}^n} Rd\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)^3} \\
&\quad - \frac{\left(\int_{\mathcal{M}^n} Rd\mu\right)\left(\int_{\mathcal{M}^n}\left(g^{\alpha\bar\beta}\frac{\partial g_{\alpha\bar\beta}}{\partial t}\right)^2 - \left|\frac{\partial g_{\alpha\bar\beta}}{\partial t}\right|^2 d\mu\right) + 2\left(\frac{d}{dt}\int_{\mathcal{M}^n} Rd\mu\right)\left(\frac{d}{dt}\int_{\mathcal{M}^n} d\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)^2}.
\end{aligned}$$

Proof. Since

$$\frac{d^2}{dt^2}\int_{\mathcal{M}^n} Rd\mu = \int_{\mathcal{M}^n}\frac{\partial^2}{\partial t^2}Rd\mu + \int_{\mathcal{M}^n} R\frac{\partial^2}{\partial t^2}d\mu + 2\int_{\mathcal{M}^n}\frac{\partial}{\partial t}R\frac{\partial}{\partial t}d\mu,$$

put the evolutions of R and μ in, we get the result for $\int_{\mathcal{M}^n} Rd\mu$.

Since

$$\frac{dr}{dt} = \frac{\left(\frac{d}{dt}\int_{\mathcal{M}^n} Rd\mu\right)\left(\int_{\mathcal{M}^n} d\mu\right) - \left(\int_{\mathcal{M}^n} Rd\mu\right)\left(\frac{d}{dt}\int_{\mathcal{M}^n} d\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)^2},$$

and

$$\begin{aligned}\frac{d^2r}{dt^2} &= \frac{\left(\frac{d^2}{dt^2}\int_{\mathcal{M}^n} Rd\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)} - \frac{\left(\frac{d^2}{dt^2}\int_{\mathcal{M}^n} d\mu\right)\left(\int_{\mathcal{M}^n} Rd\mu\right) + 2\left(\frac{d}{dt}\int_{\mathcal{M}^n} d\mu\right)\left(\frac{d}{dt}\int_{\mathcal{M}^n} Rd\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)^2} \\ &\quad + \frac{2\left(\frac{d}{dt}\int_{\mathcal{M}^n} d\mu\right)^2\left(\int_{\mathcal{M}^n} Rd\mu\right)}{\left(\int_{\mathcal{M}^n} d\mu\right)^3},\end{aligned}$$

put the evolutions of R and μ in, we get the result for r.

$\square$

Remark. In Kähler-Ricci flow, the evolutions of $\int_{\mathcal{M}^n} Rd\mu$ and r are

$$\frac{\partial}{\partial t}\int_{\mathcal{M}^n} Rd\mu = 0,$$

and

$$\frac{\partial}{\partial t}r = r^2.$$

While in our case, all the extra terms are of first-order derivatives.

§5 Further discussions

The flow considered here is the complex version of the hyperbolic geometric flow introduced by Kong and Liu [11]. Kong et al [12] proved that on Riemann surfaces, the long time existence of hyperbolic geometric flow depends on the choice of the initial velocity. We can expect that its complex version has similar property. On Calabi-Yau

manifolds, the flow can be simplified to a single complex hyperbolic Monge-Ampère equation. Note that Yau [19] used the elliptic Monge-Ampère equation to prove the famous Calabi's conjecture and Cao [2] used its parabolic version and techniques from Kähler-Ricci flow to reprove Calabi's conjecture. I hope the hyperbolic Monge-Ampère equation is also powerful to understand Kähler manifolds. We can expect this new flow is helpful to study wave phenomena in the nature especially the Einstein equation. In the future, we will study several fundamental problems on the hyperbolic Kähler-Ricci flow, for example, long-time existence, formation of singularities as well as physical applications.

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